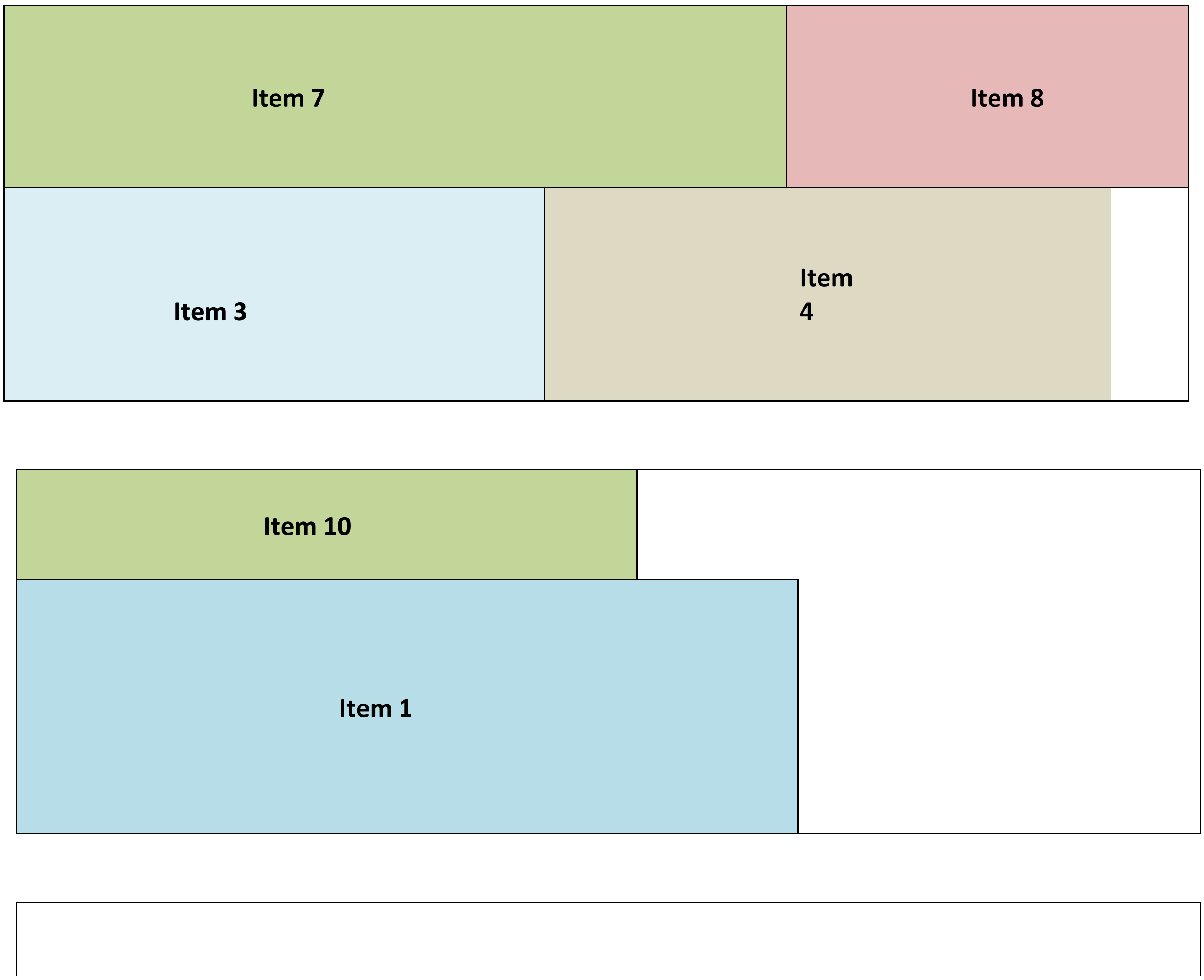
<u>Two dimensional bin packing problem</u>

The two dimensional bin packing problem is essentially a problem of fitting rectangles into minimum number of given large rectangles. Given rectangles (bins) of size W x H (where W is the width and H is the height) and given n items each width w_j and height h_j, the problem is to fit the items into minimum number of bins. We explain the problem using an example

<u>Illustration 6.9</u>

We have bins of size 15 x 10 and there are ten items to pack. The width and height of the items are (10, 7), (9, 5), (7, 5), (7, 5), (3, 5), (3, 5), (10, 4), (5, 4), (5, 4) and (8 x 3). Provide a feasible solution to the problem?

One feasible solution is shown in Figure 6.1.



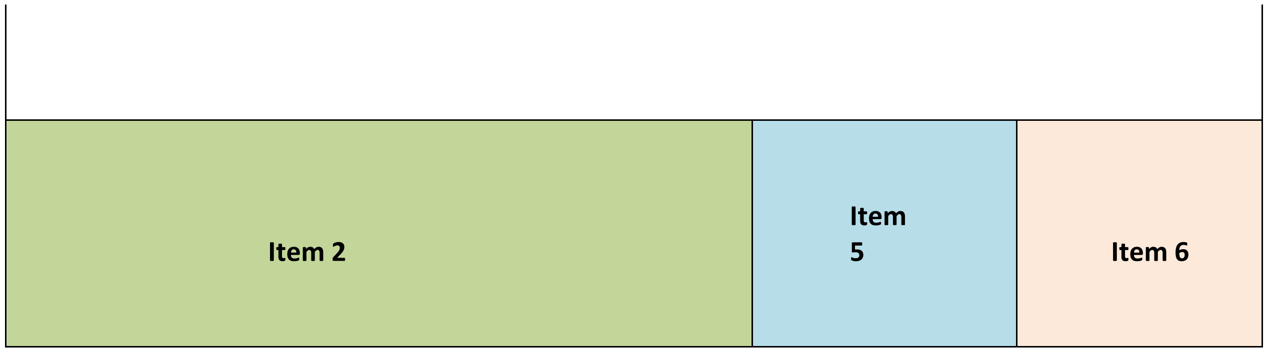


Figure 6.1 – Feasible solution

In the above solution, we use three bins. Since we wish to minimize the number of bins, we formulate an optimization problem that minimizes the number of bins. The Integer Programming formulation (Lodi et. Al, 2002) is given below:

$$Minimize \sum_{k=1}^{n} q_k$$

Subject to

$$\sum_{i=1}^{j-1} x_{ij} + y_j = 1 \text{ for j = 1,...n}$$

$$\sum_{j=i+1}^{n} w_j x_{ij} \le (W - w_i) y_i \quad \text{i = 1,...,n-1}$$

$$\sum_{k=1}^{i-1} z_{ki} + q_i = y_i \quad \text{i = 1,...,n}$$

$$\sum_{i=k+1}^{n} h_i z_{ki} \le (H - h_k) q_k \quad \text{k = 1,...,n-1}$$

Y_i, x_{ij}, q_k, z_{ki} = 0,1

In the above formulation q_k = 1 if bin k is chosen. This means that a bin is started with item k in the left bottom position. y_i = 1 if item i initializes level i. x_{ij} = 1 if item j goes to the level initialized by item i. z_{ki} = 1 if item i initializes a level in bin k.

Some of the assumptions are

1. In each level, the left most item is the tallest one
2. In each bin, bottom most level is the tallest level.
3. Items are sorted and renumbered in decreasing (non increasing) order of heights.

Illustration 6.10

We have bins of size 15 x 10 and there are ten items to pack. The width and height of the items are (10, 7), (9, 5), (7, 5), (7, 5), (3, 5), (3, 5), (10, 4), (5, 4), (5, 4) and (8 x 3). Formulate and solve a problem to find the optimal solution?

The mathematical programming formulation is as follows:
Minimize q1+q2+q3+q4+q5+q6+q7+q8+q9+q10
subject to
y1=1
x12+y2=1
x13+x23+y3=1
x14+x24+x34+y4=1
x15+x25+x35+x45+y5=1
x16+x26+x36+x46+x56+y6=1
x17+x27+x37+x47+x57+x67+y7=1
x18+x28+x38+x48+x58+x68+x78+y8=1
x19+x29+x39+x49+x59+x69+x79+x89+y9=1
x110+x210+x310+x410+x510+x610+x710+x810+x910+y10=1
9x12+7x13+7x14+3x15+3x16+10x17+5x18+5x19+8x110-5y1<=0
7x23+7x24+3x25+3x26+10x27+5x28+5x29+8x210-6y2<=0
7x34+3x35+3x36+10x37+5x38+5x39+8x310-8y3<=0
3x45+3x46+10x47+5x48+5x49+8x410-8y4<=0
3x56+10x57+5x58+5x59+8x510-12y5<=0
10x67+5x68+5x69+8x610-12y6<=0
5x78+5x79+8x710-5y7<=0
5x89+8x810-10y8<=0
8x910-10y9<=0
q1-y1=0
z12+q2-y2=0
z13+z23+q3-y3=0
z14+z24+z34+q4-y4=0
z15+z25+z35+z45+q5-y5=0
z16+z26+z36+z46+z56+q6-y6=0
z17+z27+z37+z47+z57+z67+q7-y7=0
z18+z28+z38+z48+z58+z68+z78+q8-y8=0
z19+z29+z39+z49+z59+z69+z79+z89+q9-y9=0
z110+z210+z310+z410+z510+z610+z710+z810+z910+q10-y10=0
5z12+5z13+5z14+5z15+5z16+4z17+4z18+4z19+3z110-5q1<=0
5z23+5z24+5z25+5z26+4z27+4z28+4z29+3z210-7q2<=0
5z34+5z35+5z36+4z37+4z38+4z39+3z310-7q3<=0
5z45+5z46+4z47+4z48+4z49+3z410-7q4<=0
5z56+4z57+4z58+4z59+3z510-7q5<=0

4z67+4z68+4z69+3z610-7q6<=0

4z78+4z79+3z710-8q7<=0

4z89+3z810-8q8<=0

3z910-8q9<=0

The zero one formulation to our example with ten items has 38 constraints and 110 variables. The optimal solution is given by

$q_1 = q_2 = 1$; $z_{13} = z_{27} = z_{2,10} = 1$; $y_1 = y_2 = y_3 = y_7 = y_{10} = 1$; and $x_{34} = x_{25} = x_{26} = x_{78} = x_{19} = 1$.

The above solution is interpreted as follows:

$q_1 = q_2 = 1$ indicates that there are two bins initialized by items 1 and 2. $z_{13} = z_{27} = z_{2,10} = 1$ means that item 3 initializes level 2 in bin 1 while items 7 and 10 initialize two levels in the second bin which is initialized by item 3. $x_{34} = x_{25} = x_{26} = x_{78} = x_{19} = 1$ indicate that item 4 is in the level initialized by item 3, items 5 and 6 are in the level initialized by item 2, item 8 is in the level initialized by item 7 and item 9 is in the level initialized by item 1. The solution is shown in Figure 6.2.

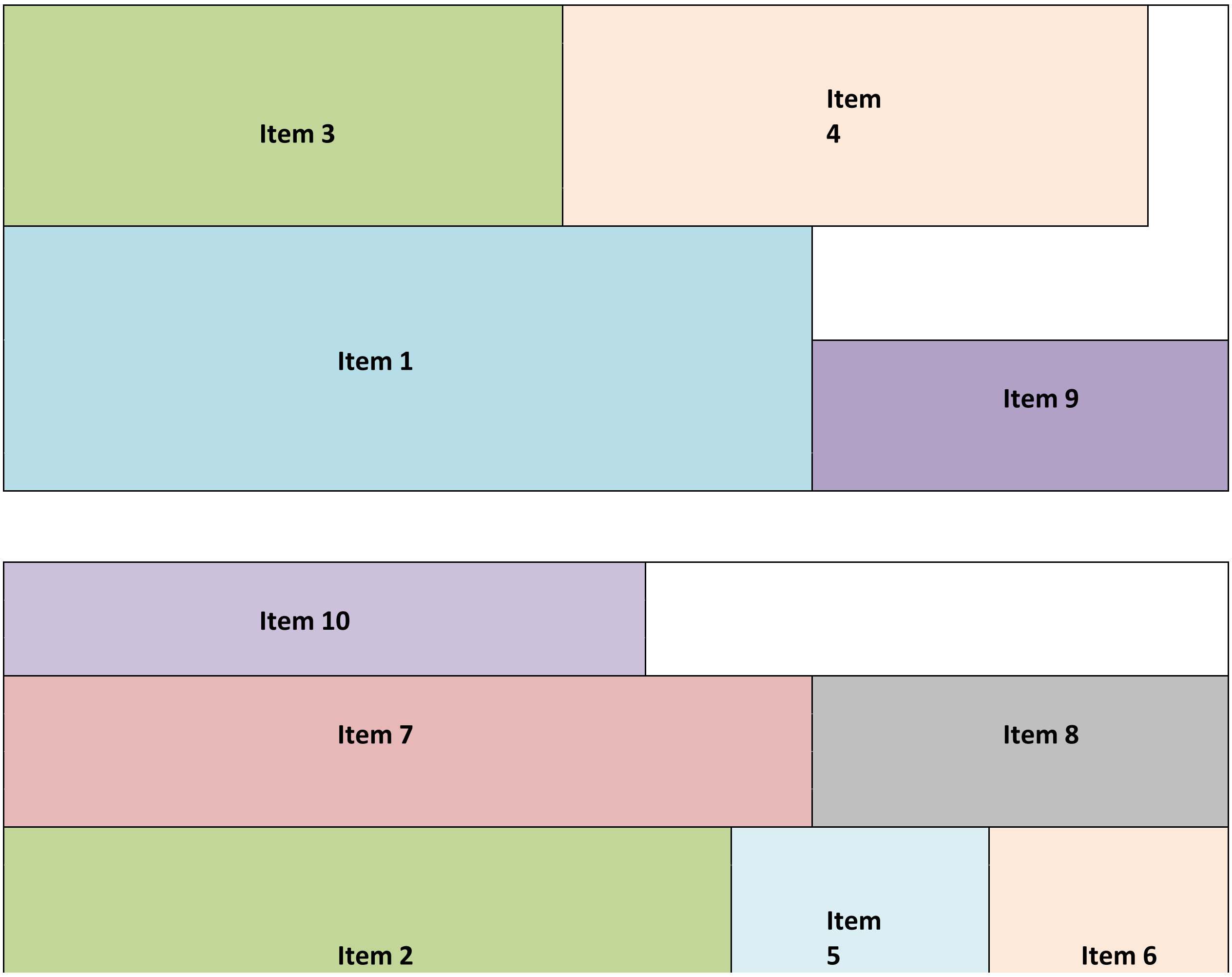


Figure 6.2 – Optimum solution

It is also necessary to note that items are not rotated. Items are placed in bins such that their widths are parallel to the width of the bin and heights are parallel to the heights. We do not rotate the items such that they can fit in. For example, if we have two items 6 x 5 and 6 x 2 and the bin size is 8 x 6, the optimum solution under the assumptions would be 2 bins. We start bin 1 with item 1 that occupies 6 x 5. The second item cannot fit into the same level because the length would be exceeded. We cannot create another level because the height would be exceeded. We need a second bin in this case.

However, if we allow rotation, we can fit both into the same bin. The second item when rotated becomes a 2 x 6 item. We start with 2 x 6 item in the first bin. The second item 6 x 5 can fit at the same level. We use the non rotation assumption and proceed to our discussion on the problem.

Two dimensional strip packing

One variation of the bin packing problem is the two dimensional strip packing problem where we assume an infinitely long (height) rectangle of width W and we wish to pack the items such that minimum height is cut. The corresponding formulation for the strip packing problem is to

$$Minimize \sum_{i=1}^{n} h_i y_i$$

Subject to

$$\sum_{i=1}^{j-1} x_{ij} + y_j = 1 \text{ for j = 1,…n}$$

$$\sum_{j=i+1}^{n} w_j x_{ij} \le (W - w_i) y_i \quad \text{i = 1,…,n-1}$$

y_i, x_{ij} = 0,1

Illustration 6.11

We have bins of size 15 x 10 and there are ten items to pack. The width and height of the items are (10, 7), (9, 5), (7, 5), (7, 5), (3, 5), (3, 5), (10, 4), (5, 4), (5, 4) and (8 x 3). Formulate and solve a 2 dimensional strip packing problem for the given situation?

The optimal solution is $y_1 = y_2 = y_3 = y_7 = y_{10} = 1$; and $x_{34} = x_{25} = x_{26} = x_{78} = x_{19} = 1$. The interpretation is the same and the minimum height is 24. This is the same as the height of 2 sheets (bins) because in the optimal solution to the bin packing problem, we have used the full height of both the sheets.

Heuristics for strip packing

Popular approximation algorithms for 1 dimensional bin packing problem have been applied to the ***2 dimensional strip packing problem.***

The next fit decreasing Height (NFDH) algorithm: Here, we pack the item left justified on the current level if it fits. Otherwise, we create a new level with the current item in it. We apply this algorithm to our first example. We create the first level with item 1, whose dimensions are (10, 7). We create the second level with item 2 whose dimensions are (9 x 5). We create level 3 with the third item (7 x 5). Item 4 fits in this level. Items 5 and 6 fit into the next level. Items 7 and 8 make the next level and items 9 and 10 make the final level. The height of the strip is the sum of the heights of each level, which is 7 + 5 + 5 + 5 + 4 + 4 = 30. If the heights are normalized to 1, it has been shown that this algorithm for an instance I has NFDH (I) ≤ 2 OPT (I) + 1.

The First fit decreasing Height (FFDH) algorithm: Here, we pack the item left justified on the first level where it fits. Otherwise, we create a new level with the current item in it. We apply this algorithm to our first example. We create the first level with item 1, whose dimensions are (10, 7). We create the second level with item 2 whose dimensions are (9 x 5). We create level 3 with the third item (7 x 5). Item 4 fits in this level. Items 5 and 6 fit in levels 1 and 2 respectively. Items 7 and 8 make the next level and items 9 and 10 make the final level. The height of the strip is the sum of the heights of each level, which is 7 + 5 + 5 + 4 + 4 = 25. If the heights are normalized to 1, it has been shown that this algorithm for an instance I has NFDH (I) ≤ 1.7 OPT (I) + 1.

The Best fit decreasing Height (BFDH) algorithm: Here, we pack the item left justified on the level where it fits and where the horizontal residue is minimum. Otherwise, we create a new level with the current item in it. We get the same solution as FFDH for our example.

The bottom left (BL) algorithm: Here items are sorted by non decreasing widths and are placed in the lowest possible position left justified. It has been shown that this algorithm for an instance I has BL (I) ≤ 3 OPT (I).

It is possible to extend the strip packing algorithms to the 2 dimensional bin packing algorithm. We can apply the FFDH algorithm for the strip packing problem to get the levels and apply the First Fit Decreasing heuristic (FFD) to pack the items at each level. This heuristic is called the Hybrid first fit (HFF) heuristic. If the heights are normalized to 1, for an instance I, HFF (I) ≤ 17/8 OPT (I) + 5.

Illustration 6.12

We have bins of size 15 x 10 and there are ten items to pack. The width and height of the items are (10, 7), (9, 5), (7, 5), (7, 5), (3, 5), (3, 5), (10, 4), (5, 4), (5, 4) and (8 x 3). Apply the Bottom left (BL) algorithm to this data. Provide heuristic solutions to the problem?

We apply the BL algorithm to the given data. The sorted order of items is (10, 7), (10, 4), (9, 5), (8 x 3), (7, 5), (7, 5), (5, 4), (5, 4), (3, 5) and (3, 5).

We create the first level with item 1, whose dimensions are (10, 7). We create the second level with item 2 whose dimensions are (10 x 4). We create level 3 with the third item (9 x 5). We create a fourth level with the item (8 x 3). The fifth item (7 x 5) goes with the level (8 x 3) increasing its height to 5. We create another level for the sixth item (7 x 5). Item 7 (5 x 4) fits in level 1 and item 8 (5 x 4) fits in level 2. Items 9 and 10 each with (3 x 5) fit in the level (9 x 5). The height of the strip is the sum of the heights of each level, which is 7 + 4 + 5 + 5 + 5 = 26. The solution is shown in Figure 6.3.

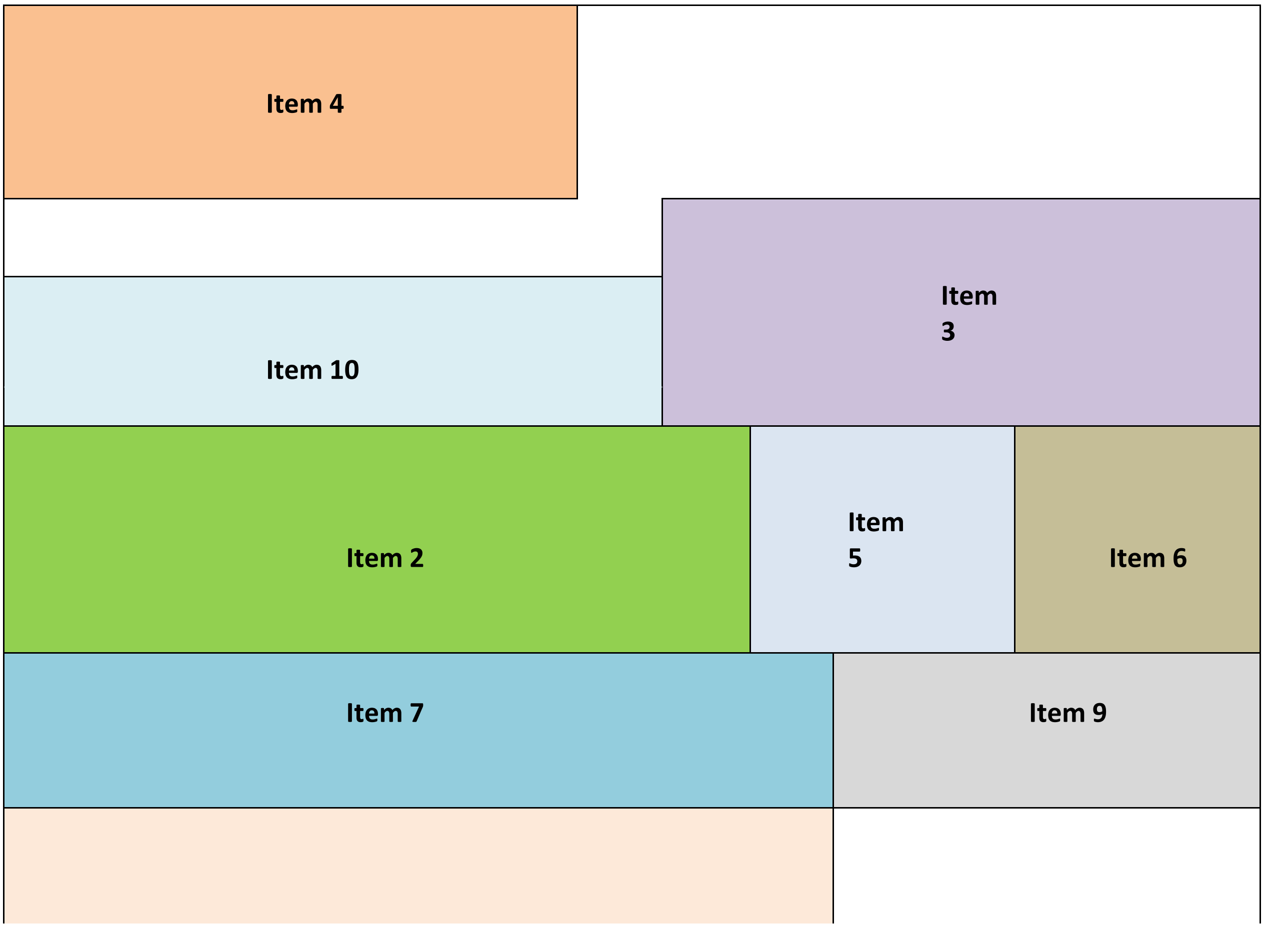


Figure 6.3 – Solution using BL algorithm

If we apply the FFDH algorithm for the strip packing problem, we get five levels with height 7, 5, 5, 4, 4. Since the bin height is 12, we can apply the First Fit Decreasing heuristic (FFD) for the 1 dimensional bin packing problem to get three bins that fit 7 + 5, 5 + 4 and 4 respectively. We require 3 bins based on this heuristic. We have seen that the optimum solution requires 2 bins.

Three dimensional bin packing

The three dimensional bin packing problem is to pack cuboids into three dimensional bins which are large cuboids. We are given a set of 3d rectangular objects where w_j, h_j and d_j, represent the width, height and depth of item j. We pack these into identical bins with width W, depth D, and height H using minimum bins. It is also assumed that during the packing, the smaller bins are packed lengthwise along the length L and so on.

The 3-d bin packing problem can be further classified to include:

Single item single bin
Multiple item single bin
Single item, multiple bins
Multiple items, multiple bins

Though mathematical programming formulations to minimize the number of bins exist, it is customary to use heuristic algorithms to solve this problem, considering the large number of variables and constraints in the formulation.

Several mathematical programming formulations exist for the 3 dimensional bin packing problem. We present a mixed integer linear programming formulation from Hifi et. Al (2010).

Let (x_i, y_l, z_i) represent the left-bottom-back coordinate of item i, while the left-bottom back coordinate of the bin is taken as (0, 0, 0). Let q_i be the bin to which item i is assigned. Let l_{ij} = 1 if item i is to the left of item j (= 0 otherwise). Let u_{ij} = 1 if item i is under item j (= 0, otherwise). Let b_{ij} = 1 if item i is to the back of item j (= 0, otherwise). Let C_{ij} = 1 if items i and j are in different bins (= 0, if they are in the same bin). Let q be the number of bins used. Let Q be a known number of bins into which it is possible to pack all the items. The value given to Q is an upper bound on q. We can assume that Q = n (the number of items). The value of Q can be brought down by using a heuristic solution.

The objective function is to Minimize q
Subject to

$l_{ij} + l_{ji} + u_{ij} + u_{ji} + b_{ij} + b_{ji} + c_{ij} + c_{ji} = 1 \quad i < j = 1, \dots, n$

$x_i - x_j + W(l_{ij} - c_{ij} - c_{ji}) \le W - w_i, \quad i \ne j = 1, \dots, n$

$y_i - y_j + H(u_{ij} - c_{ij} - c_{ji}) \le H - h_i, \quad i \ne j = 1, \dots, n$

$z_i - z_j + D(b_{ij} - c_{ij} - c_{ji}) \le B - b_i, \quad i \ne j = 1, \dots, n$

$(Q-1)(l_{ij} + l_{ji} + u_{ij} + u_{ji} + b_{ij} + b_{ji}) + q_i - q_j + q_1 c_{ij} \le Q - 1 \quad i \ne j = 1, \dots, n$

$0 \le x_i \le W - w_i \quad i = 1, \dots, n$

$0 \le y_i \le H - h_i \quad i = 1, \dots, n$

$0 \le z_i \le D - d_i \quad i = 1, \dots, n$

$1 \le q_i \le q \le Q \quad i = 1, \dots, n$

$l_{ij}, u_{ij}, b_{ij}, c_{ij} = 0, 1 \quad i \ne j = 1, \dots, n$

$q_i integer$

The objective function minimizes the number of used bins. The first constraint ensures that two items with either go to the same bin or will go to different bins. It also ensures that if two items go to the same bin they are either to the left of each other or behind each other or under each other. It also prevents situations where item i is to the left of j and item j is to the left of itemi and so on.

The second set of constraints ensures that if item i is placed to the left of j, there is sufficient length to accommodate the item. Similarly the third and fourth sets of constraints take care of sufficient width and height available for items to be packed in the same bin. These constraints become redundant if items i and j are assigned to different bins.

The fifth set of constraints creates new bins. If items belong to different bins $q_i - q_j$ has to be 1 resulting in the creation of a new bin. If they are in the same bin, this constraint becomes redundant.

The sixth seventh and eighth set of constraints ensures no overlapping. The ninth set of constraints ensures that each item is assigned to a bin. It also ensures that we have fewer or equal to q_1 bins. It also links the variable q (minimum number of bins) to q_i and q_1. We also have the binary and integer variables that are defined.

Illustration 6.13

Consider W = 9, H = D = 7. Consider three items with dimensions (5, 4, 5), (2, 7, 7) and (2, 6, 6). Also solve for W = D = H = 7 for the same three items.

We formulate for W = 9, D = H = 7 and (5, 4, 5), (2, 7, 7) and (2, 6, 6). The formulation is

Minimize q

subject to

$x_1 \le 4$; $x_2 \le 7$; $x_3 \le 7$

$y_1 \le 3$; $y_2 \le 0$; $y_3 \le 1$

$z_1 \le 2$; $z_2 \le 0$; $z_3 \le 1$

$q_1 \ge 1$; $q_2 \ge 1$; $q_3 \ge 1$

$q_1 - q \le 0$; $q_2 - q \le 0$; $q_3 - q \le 0$; $q \le 2$

$l_{12} + l_{21} + u_{12} + u_{21} + b_{12} + b_{21} + c_{12} + c_{21} = 1$

$l_{13} + l_{31} + u_{13} + u_{31} + b_{13} + b_{31} + c_{13} + c_{31} = 1$

$l_{23} + l_{32} + u_{23} + u_{32} + b_{23} + b_{32} + c_{23} + c_{32} = 1$

$x_1 - x_2 + 9l_{12} - 9c_{12} - 9c_{21} \le 4$

$x_1 - x_3 + 9l_{13} - 9c_{13} - 9c_{31} \le 4$

$x_2 - x_1 + 9l_{21} - 9c_{21} - 9c_{12} \le 7$

$x_2 - x_3 + 9l_{23} - 9c_{23} - 9c_{32} \le 7$

$x_3 - x_1 + 9l_{31} - 9c_{31} - 9c_{13} \le 7$

$x_3 - x_2 + 9l_{32} - 9c_{32} - 9c_{23} \le 7$

$y_1 - y_2 + 7u_{12} - 7c_{12} - 7c_{21} \le 3$

$y_1 - y_3 + 7u_{13} - 7c_{13} - 7c_{31} \le 3$

$y_2 - y_1 + 7u_{21} - 7c_{21} - 7c_{12} \le 0$

$y_2 - y_3 + 7u_{23} - 7c_{23} - 7c_{32} \le 0$

$y_3 - y_1 + 7u_{31} - 7c_{31} - 7c_{13} \le 1$

$y_3 - y_2 + 7u_{32} - 7c_{32} - 7c_{23} \le 1$

$z_1 - z_2 + 7b_{12} - 7c_{12} - 7c_{21} \le 2$

$z_1 - z_3 + 7b_{13} - 7c_{13} - 7c_{31} \le 2$

$z_2 - z_1 + 7b_{21} - 7c_{21} - 7c_{12} \le 0$

$z_2 - z_3 + 7b_{23} - 7c_{23} - 7c_{32} \le 0$

$z_3 - z_1 + 7b_{31} - 7c_{31} - 7c_{13} \le 1$

$z_3 - z_2 + 7b_{32} - 7c_{32} - 7c_{23} \le 1$

$l_{12} + l_{21} + u_{12} + u_{21} + b_{12} + b_{21} + q_1 - q_2 + 2c_{12} \le 1$

$l_{13} + l_{31} + u_{13} + u_{31} + b_{13} + b_{31} + q_1 - q_3 + 2c_{13} \le 1$

$l_{21} + l_{12} + u_{21} + u_{12} + b_{21} + b_{12} + q_2 - q_1 + 2c_{21} \le 1$

$l_{23} + l_{32} + u_{23} + u_{32} + b_{23} + b_{32} + q_2 - q_3 + 2c_{23} \le 1$

$l_{31} + l_{13} + u_{31} + u_{13} + b_{31} + b_{13} + q_3 - q_1 + 2c_{31} \le 1$

$l_{32} + l_{23} + u_{32} + u_{23} + b_{32} + b_{23} + q_3 - q_2 + 2c_{32} \le 1$

We have used Q = 2 in this formulation. We have 25 variables and 43 constraints. In addition we have the binary restriction on many variables and integer restriction on few variables. These are not shown explicitly in the formulation.

The optimum solution to the formulation is given by $q_1 = q_2 = q_3 = q = 1$. $x_1 = 2$, $x_3 = 7$ and $l_{21} = l_{13} = l_{23} = 1$.

The solution packs all the three items into a **single bin**. Item 2 is to the left of item 1, item 1 is to the left of item 3 and item 2 is to the left of item 3. The coordinates for the three items are (2, 0, 0), (0, 0, 0) and (7, 0, 0) respectively. Item 2 is placed in the bottom left corner starting with (0, 0, 0). Item 1 is placed next starting at (2, 0, 0) and then item 3 starting at (7, 0, 0). The widths and heights are all less than or equal to 7.

We formulate for W = D = H = 7 and (5, 4, 5), (2, 7, 7) and (2, 6, 6). The formulation has the same number of variables and constraints. The coefficients in some of the constraints change because W = 7. We use Q = 2. The optimum solution is given by $q_1 = q_3 = q = 2$; $q_2 = 1$, $x_1 = 2$ and $c_{21} = l_{31} = c_{23} = 1$.

Here we use 2 bins. It is obvious that the sum of the lengths exceeds 7 and hence the three items cannot be packed lengthwise at the same level. The widths and heights do not allow a second level to be created. Items 1 and 3 go to the second bin while item 2 goes to the first bin. Item 2 is placed in the first bin at (0, 0, 0). Item 3 is placed in the second bin at (0, 0, 0) and item 1 is placed next at (2, 0, 0). We also have the corresponding c_{ij} values taking 1.

Illustration 6.14

Consider W = D = H = 12. Consider five items with dimensions (8, 6, 4), (6, 5, 8), (4, 8, 3), (4, 5, 7) and (6, 5, 5). Assume Q = 2

We have n = 5. We have 20 variables each of l_{ij}, b_{ij}, u_{ij} and c_{ij}. We have 5 q_i variables, x_i, y_i and z_i variables and q. We have 101 variables. Constraint set 1 has 10 constraints while sets 2, 3 and 4 have 20 constraints each. There are 20 constraints in set 5. There are 25 more constraints for bounds on x_i, y_i, z_i and q_i. There are 116 constraints.

The optimum solution is given by $l_{24} = l_{25} = l_{34} = l_{35} = u_{12} = u_{54} = b_{31} = b_{32} = b_{41} = b_{51} = 1$; $q_1 = q_2 = q_3 = q_4 = q_5 = q = 1$; $x_1 = 4$, $x_4 = 6$, $x_5 = 6$, $y_2 = 6$, $y_4 = 5$, $z_1 = 8$, $z_2 = 4$.

All items fit into one bin. The coordinates are (4, 0, 8), (0, 6, 0), (0, 0, 0), (6, 5, 0) and (0, 5, 0). From the coordinates we can see that we pack with item 3 in the bottom left back corner.

Item 5 is packed starting at (6, 0, 0) at the base. There is a gap between items 3 and 5. Item 4 is packed in front of item 5 in the base starting at (6, 5, 0). Items 1 and 2 are on top of the three items kept on the base. Item 1 is placed at (4, 0, 8) and is a 8x6x4 cuboid. It can be assumed to be pasted from the top right corner. Item 2 starts at (0, 6, 4) and is a 6x5x8 cuboid. It can be assumed to be pasted to the top. We can verify the l_{ij}, u_{ij} and b_{ij} values that are set to 1. Part of the solution is shown in Figure 6.4

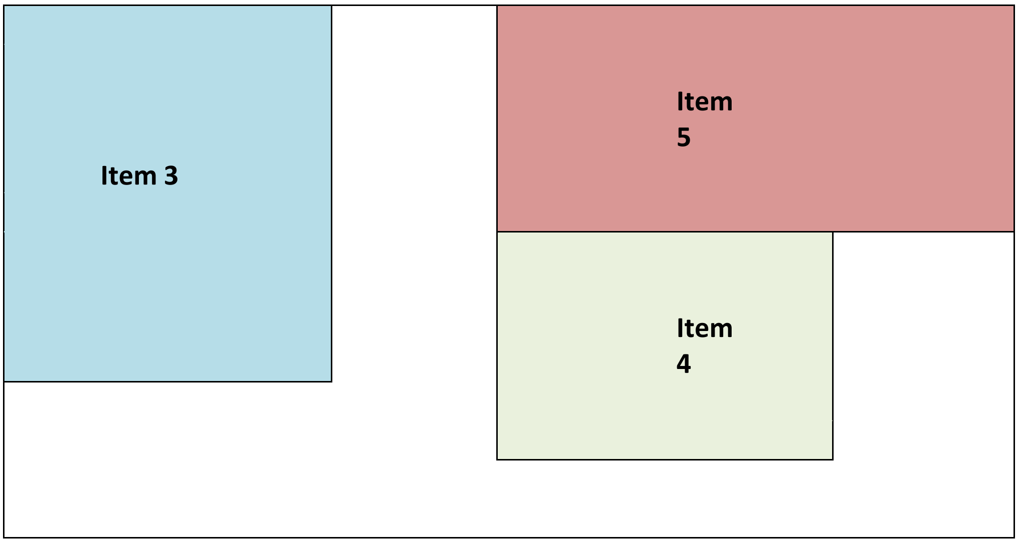


Figure 6.4 – Part of the solution

The above formulation is able to fit smaller cuboids into larger cuboids. The above solution however does not place the items one above the other. There is a gap between items 1 and 5 even though item 1 is placed above item 5.

Heuristics for 3 dimensional bin packing

Several heuristics are available to solve the three dimensional bin packing problem. The reader may refer to Lodi et. al (2002) for some of them. Most of them are based on reducing the three dimensional problem to a two dimensional or one dimensional bin packing problem based on certain criteria.

We illustrate three simple heuristic algorithms for the three dimensional bin packing problem. The algorithms are as follows:

Heuristic 1 (H1)

1. Sort the items according to non increasing order of height.
2. Pack them into the latest bin based on w x d from the bottom left top corner. Create a new level if required.
3. Create a new bin and repeat step 2.
4. Repeat 2 and 3 till all items are packed

Heuristic 2 (H2)

1. Sort items according to non increasing order of w x d.
2. Pack them into the latest bin based on w x d from the bottom left top corner. Create a new level if required.
3. Create a new bin and repeat step 2.
4. Repeat 2 and 3 till all items are packed

Heuristic 3 (H3)

1. Sort the items according to non increasing order of height.
2. Solve a one dimensional bin packing problem to group them into bins.
3. Pack items according to the groups into the latest bin based on w x d from the bottom left top corner. Place the items within a group one above the other.
4. Create a new bin and repeat step 3.
5. Repeat 2 and 3 till all items are packed.

Illustration 6.15

Consider W = D = H = 12. Consider five items with dimensions (8, 6, 4), (6, 5, 8), (4, 8, 3), (4, 5, 7) and (6, 5, 5). Solve the three dimensional bin packing problem using H1, H2 and H3?

We illustrate heuristic H1. The sorted order according of items to height is 2, 4, 5, 1, 3. We start bin 1 and place item 2 in the bottom left back corner. The coordinate is (0, 0, 0). We place item 4 next to item 2. The coordinate of the bottom, left back corner of item 4 is (6, 0, 0). We place item 5 in front of item 2. The coordinates for item 5 are (5, 0, 0). Item 1 cannot be placed on the base and has to be placed at a higher level (on top of an already placed item). Since item 1 has a height of 6 units, it can be placed on top of item 5. The coordinates of the bottom, back left corner of item 1 is (0, 6, 5). Item 3 cannot be placed in bin 2. It cannot be placed by the side of 5 or 1 nor can it be placed on top of 2 or 4. We need a second bin to place item 3. This heuristic gives us a solution with 2 bins.

We illustrate heuristic H2. The sorted order of items according to decreasing values of w x d is 1, 3, 2, 5, 4. We start the first bin with item 1 with coordinates (0, 0, 0). We place item 3 next to item 1. The coordinates are (8, 0, 0). We place item 2 in front of item 1. The coordinates are (0, 6, 0). Item 5 cannot be placed by the side of item 2 because the width is exceeded. Item 5 therefore is placed above item 1. The coordinates are (0, 0, 4). Item 4 is placed above item 3. The coordinates are (6, 0, 3). We require one bin only.

We illustrate heuristic H3. The sorted order according to non decreasing heights is 2, 4, 5 1, 3. We solve a one dimensional bin packing problem to get three groups {2, 1}, {4, 5}, {3}. We start by placing item 2 in the bottom left back corner. The coordinates are (0, 0, 0). Item 1 is placed above item 2 (as a result of bin packing based on heights). Item 1 has coordinates (0, 0, 8). Item 4 is taken next and can be placed to the right of item 2 or in front of item 2. We place it in front of item 2. The coordinates are (0, 6, 0). We have used this coordinate taking into consideration that item 1 has been placed already. Item 5 is placed above item 4 (belongs to the same bin). The coordinates are (0, 6, 7). Item 3 is placed next to item 2 or can be placed from the right, bottom back corner. The coordinates are (8, 0, 0). We require only one bin.